

Rational Function Integration

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Abstract

The derivative of any rational function is a rational function. An algorithm and decision procedure for finding the rational function anti-derivative of a rational function is presented.

1. Rational Function Differentiation

Let

$$y(x) = \prod_{j \neq 0} p_j(x)^j \quad (1)$$

be a rational function of x where the primitive polynomials $p_j(x)$ are square-free and mutually relatively prime.

The derivative of $y(x)$ is

$$\frac{\partial y}{\partial x}(x) = \sum_{j \neq 0} j p_j(x)^{j-1} p'_j(x) \prod_{0 \neq k \neq j} p_k(x)^k \quad (2)$$

Lemma 1. *Given square-free and relatively prime primitive polynomials $p_j(x)$, the expression $\sum_j j p'_j(x) \prod_{k \neq j} p_k(x)$ has no factors in common with $p_j(x)$.* ■

Assume that the expression has a common factor $p_h(x)$ such that:

$$p_h(x) \left| \sum_j j p'_j(x) \prod_{k \neq j} p_k(x) \right.$$

$p_h(x)$ divides all terms for $j \neq h$. Because it divides the whole sum, $p_h(x)$ must divide the remaining term $h p'_h(x) \prod_{k \neq h} p_k(x)$. From the given conditions, $p_h(x)$ does not divide $p'_h(x)$ because $p_h(x)$ is square-free; and $p_h(x)$ does not divide $p_k(x)$ for $k \neq h$ because they are relatively prime.

2. Rational Function Integration

Separating square and higher factors of equation (2):

$$\frac{\partial y}{\partial x}(x) = \left[\prod_j p_j(x)^{j-1} \right] \left[\sum_j j p'_j(x) \prod_{k \neq j} p_k(x) \right] \quad (3)$$

There are no common factors between the sum and product terms of equation (3) because of the relatively prime condition of equation (1) and because of Lemma 1. Hence, this equation cannot be reduced and is canonical.

Split equation (3) into factors by the sign of the exponents, giving:

$$\frac{\partial y}{\partial x}(x) = \frac{\prod_{j>2} p_j(x)^{j-1}}{\prod_{j<0} p_j(x)^{1-j}} \overbrace{\sum_j j p'_j(x) \prod_{k \neq j} p_k(x)}^{\mathbf{L}} \quad (4)$$

The denominator is $\prod_{j<0} p_j(x)^{1-j}$. Its individual $p_j(x)$ can be separated by square-free factorization. The $p_j(x)$ for $j > 2$ can also be separated by square-free factorization of the numerator. Neither $p_2(x)$ nor $\sum_j j p'_j(x) \prod_{k \neq j} p_k(x)$ have square factors; so square-free factorization will not separate them. Treating $p_2(x)$ as 1 lets its factor be absorbed into $p_1(x)$. Note that $p_0(x) = 1$; also $p_j(x) = 1$ for exponents j which don't occur in $\partial y / \partial x$. All the $p_j(x)$ are now known except $p_1(x)$. Once $p_1(x)$ is known, $y(x)$ can be recovered. Let polynomial \mathbf{L} be the result of dividing the numerator of $\partial y / \partial x$ by $\prod_{j>2} p_j(x)^{j-1}$.

$$\overbrace{\sum_j j p'_j(x) \prod_{k \neq j} p_k(x)}^{\mathbf{L}} = \overbrace{\sum_{j \neq 1} j p'_j(x) \prod_{1 \neq k \neq j} p_k(x)}^{\mathbf{M}} p_1(x) + p'_1(x) \overbrace{\prod_{k \neq 1} p_k(x)}^{\mathbf{N}}$$

Because they don't involve $p_1(x)$, polynomials \mathbf{M} and \mathbf{N} can be computed from the square-free factorizations of the numerator and denominator. The polynomial $p_1(x)$ can be constructed by the following procedure where $\mathbf{deg}(p)$ is the degree of x in polynomial p , $\mathbf{coeff}(p, w)$ is the coefficient of the x^w term of p , and $\mathbf{lc}(p)$ is the leading coefficient of polynomial p . Let \mathbf{A} , \mathbf{C} , and \mathbf{R} be rational expressions. Only the numerators of \mathbf{A} and \mathbf{R} contain powers of x . Starting from polynomials \mathbf{L} , \mathbf{M} , and \mathbf{N} :

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A := 0;
R := L;
while ((w := min(deg(R)-deg(M), deg(R)-deg(N)+1)) >= 0) do
  C := lc(R) / (coeff(M, deg(R)-w) + w*coeff(N, deg(R)-w+1));
  A := A + C*x^w;
  R := R - C*(M*x^w + w*N*x^(w-1));

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At the end of this process, if $\mathbf{R} = 0$, then $p_1(x)$ is the numerator of \mathbf{A} and $y(x) = \prod_j p_j(x)^j$ divided by the denominator of \mathbf{A} . Otherwise the anti-derivative is not a rational function.